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COLOURING THE DISCRETIZATION GRAPHS ARISING IN THE MULTIGRID METHOD

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Abstract—We formulate aims and conditions of graph colourings suitable for the (triangulation) graphs arising in the finite element-based multigrid method, review graph-theoretical results, describe linear-time 6-, 5- and 4-colour algorithms and give results of numerical experiments.

1. INTRODUCTION, TERMINOLOGY

As is well known, the very fast convergence of the difference-based multigrid algorithm for the numerical solution of regular self-adjoint second order elliptic two-dimensional boundary value problem is—among others—connected with certain orderings of the unknowns during the Gauss-Seidel iteration of the smoothing step, see [1, 2] (for general expositions of multigrid algorithms see [2, 3]).

These orderings can be interpreted as "colourings" (like red-black, 4-colour) of the corresponding (discretization) graphs in the usual sense (for graph theory terminology and important results we refer to [4, 5]): a regular colouring of the nodes of a graph is a partition of the nodes into disjoint sets in such a way that nodes of the graph, which are neighbours there (connected by an edge), are in different sets (coloured differently).

We speak about "nodes" since this word is common to both graphs and discretizations; synonymously we use "unknowns," having in mind the solution of a single self-adjoint elliptic partial differential equation where to every node of the discretization grid there is attached just one unknown.

The "discretization graphs" mentioned are in the narrow sense the graphs of the (symmetric) matrices of the discretization [6, 7], on a fixed multigrid level greater one. (The first level consists of only a few nodes and is therefore of less interest.)

Considering in what follows discretization graphs arising from linear triangular finite elements and conforming refinement by subdivision of every triangle into four similar ones, we follow the finite element approach usual for multigrid [2, 3, 8]. Here it is more appropriate to identify the graphs considered with the triangular grids used.

For nonself-adjoint problems, often the matrices remain symmetric by structure, and then the same (undirected) graphs can be used as for self-adjoint problems. However, for equations with dominating first derivatives it is more effective to use special colourings which reflect the distinguished directions, see Point 5.2 below.

The speciality of the multigrid discretization graphs consists in the presence of "new" and "old" nodes on a fixed level (different from the first level) where the old nodes correspond to the corners of the triangles before refinement, (i.e., to the triangles of the coarse level); the new nodes are the midpoints of the coarse level triangle sides, i.e., the corners of the triangles after the refinement—minus the old nodes, see Figure 1.

In other words, for all levels (usually 2–7 levels) these graphs form a sequence of interrelated graphs: the node sets are subsets of the node set of any later graph of the sequence, however, the

The starting point for the review on graph colouring was a consultation given by Prof. L. Lovász, Budapest.

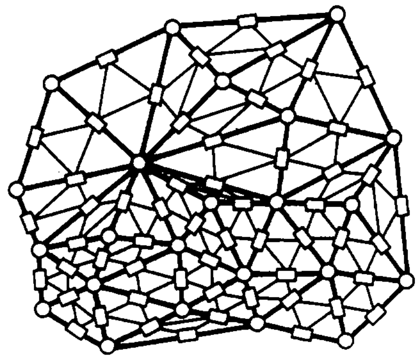

Figure 1. o old points, □ new points, —refinement.

edge sets are not subsets of each other, since in the mentioned refinement process, every edge is divided into two new edges by introducing a new node in the midpoint of the edge; between the new nodes, additional edges are introduced by the refinement of the coarse level triangles.

A systematical investigation about possibilities why and how to colour discretization graphs, including graphs different from or more general than those of standard two-dimensional difference approximations on rectangular grids seems to be lacking; for rectangular grids see [9–12].

2. AIMS AND CONDITIONS

We see several possibilities for the use of colourings. First, there is the hope to find—besides the red-black and the 4-colourings of rectangular grids—further discretization graphs with certain symmetries, such colourings which assure very fast convergence. In this respect, however, we doubt that progress is possible, see 5.1 below.

Second, the colourings as orderings of the unknowns (= nodes) into disjoint sets, by the independency property of the nodes belonging to one set (in no set are there nodes which are neighbours on the graph), represent the basis for vectorization and parallelization of iteration methods, see, e.g., [11, 12]: to the sets of equally coloured unknowns there correspond blocks of the matrix. Performing say the Gauss-Seidel step for one block, every unknown may be computed in parallel by another processor if all processors have, at the same time, access to all other blocks. Alternatively, all unknowns of one colour may be computed in vector mode.

Here it is important to remark that it is *not* the exclusive aim to colour with minimal number of colours; the number of colours is of less interest since it gives only the number of steps when the block iteration cycles over all colours (either performing vectorization or parallelization). The aim is rather to dispose of a *spectrum of colourings* so as to select that one which is best suited to the computer configuration used, and for this the number of nodes in every colour set is of more interest.

For a computer with several processors working in parallel it is of interest that the number of nodes in every colour set is divisible by the number of processors. For special processors there might arise further divisibility conditions; e.g., in case of vector processors with fixed length of the vector registers. For one vector processor without fixed vector length, it looks desirable to have a minimal number of colours since then, in the mean, the vector step will work on vectors of greater length. In any case, it is of interest to have almost uniform colourings, i.e., with nearly equal numbers of nodes belonging to every colour. All these colourings should satisfy a *first side condition*: they should be obtainable in a computing time which linearly grows with the number of nodes. Otherwise they would be more expensive than the multigrid algorithm itself.

There are some natural reasons to consider only colourings satisfying a *second side condition*: all old nodes are coloured first and get colour number 1. Indeed, this corresponds to the data structure usual for the triangular finite element-based multigrid algorithm [13]; if the old nodes would be coloured last, and straight injection be selected as restriction (this restriction is theoretically not founded [2, p. 77] but rather effective for the mentioned class of elliptic problems), then the multigrid algorithm would stop immediately on the initial approximation of the numerical solution. We remark that the usual 2-colouring (i.e., red-black ordering) and 4-colouring of rectangular graphs satisfy both side conditions.

For the investigation of problems connected with parallelization and vectorization of the multigrid algorithm, we refer to [14, 15] and remark that for our considerations it is not decisive that the discretized partial differential equations mentioned in the Introduction are solved by the multigrid method. It is decisive only in that at least one conform refinement step has been performed on an original triangular grid to obtain the grid under consideration.

3. REVIEW OF GRAPH COLOURING RESULTS

For the planar graphs considered in the Introduction, the following results [4, 5, 16]—which seem to be not widely known in the discretization community—are of interest.

PROPOSITION 1. *A graph can be 2-coloured (coloured with two colours) iff all circuit numbers are even.* ∎

(The circuit number is the number of edges around an elementary mesh of the graph; for the graph of a rectangular grid this assures the existence of the red-black ordering.)

PROPOSITION 2. (Heawood 1898) *A triangulation graph can be 3-coloured if all degrees are even. This condition is also necessary if one adds: or the graph is a subgraph of such a graph [17].* ∎

(The degree of a node is the number of edges connected to it; in a triangulation graph every elementary mesh is a triangle. For instance the coverings of parts of the plane by regular triangles can be 3-coloured; however, with this colouring the condition about the colour of the old nodes is—apart from trivial cases—not satisfied, as can be seen immediately.)

For a general planar graph the question whether it can be 3-coloured is NP complete and this holds even then if the maximal degree is known to be not greater than 4.

PROPOSITION 3. *Any planar graph can be 4-coloured (Koch, Appel, 1976, see [18]).* ∎

However, here the first side condition, in general, will not be satisfied. (B. Toft expresses his belief that the Koch-Appel-Haken approach corresponds to a polynomial-time colouring algorithm of high polynomial degree.)

REMARK. (see [5, v. II], footnote on p. 257). Due to the theorem of Tait, the 4-colour theorem for the nodes is equivalent to the 3-colour theorem for the edges (in colouring edges, the colouring is called regular, if all edges of the same elementary mesh are differently coloured). Therefore, colouring the old nodes with colour 1 and applying the 3-colour theorem to the edges of the coarse grid graphs, these latter can be coloured using colours 2 to 4. Colouring these edges, their midpoints (our new nodes) are coloured, too, and this defines just a regular 3-colouring of the new nodes. ∎

In other words, there is a 4-colouring of the graphs considered which satisfies the second side condition.

PROPOSITION 4. *There are linear-time algorithms for 5-colouring of general planar graphs [19,20].* ∎

PROPOSITION 5. (P. Unger, 1986, see [16]) *Let be* $G := \{x_j\}_{j=1,\dots,n}$ *the node set of the (planar) graph and* $x_{k(i)}$ *a node with minimal degree in* $G - \{x_j\}_{j=i+1,n}$, $i = n, n-1, \dots, 1$, *where* $\{x_j\}_{j=n+1,n}$ *is empty. Now take the sequence* $x_{k(1)}, x_{k(2)}, \dots$ *(this is the smallest last ordering) and colour these nodes sequentially using always the smallest possible colour. Then a 6-colouring results.* ∎

The existence question of (nearly) uniform colourings is completely answered by the following result.

THEOREM. (Hajnal, Szemerédi [21]) *A graph (not necessarily planar, without loops and multiple edges) of maximal node degree* m *can be coloured "uniformly" for any number* $l > m$ *of colours.*

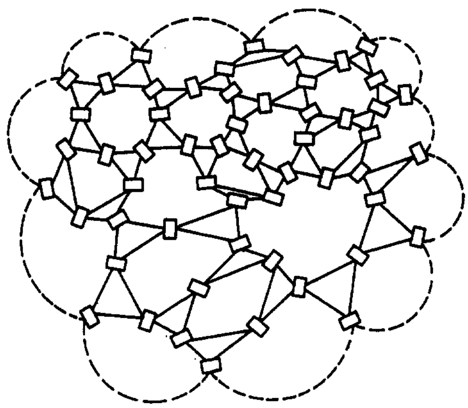

Figure 2.

Here "uniformly" is to be understood in the following (best possible) sense: Let n be the number of nodes and $n = lk + r$ (where $0 \leq r < l$), then the set V of all nodes can be subdivided into l disjunct sets A_i, $V = A_1 \cup \cdots \cup A_l$, satisfying

a) $|A_i| = k + 1,\ i = 1, \ldots, r;\ |A_i| = k,\ i = r + 1, \ldots, l$;

b) no A_i contains a pair of nodes connected by an edge. ∎

Applied to the usual discretization graphs for the two-dimensional problems considered with degrees not exceeding 6, this theorem assures the existence of "uniform" colourings with 7 or more colours.

Of course, the old nodes may be of any degree where high degrees can occur only at the nodes of the coarsest grid. Therefore, it makes sense to apply the theorem to the graphs remaining after deletion of the old nodes thus ensuring existence of a "uniform" 5-colouring of the new nodes.

Assuming this colouring being given (an algorithm seems to be unknown), in one colour set of the colours 2 to 6 there is then asymptotically about 3/20 of the (new) nodes, whereas 1/4 of the nodes have colour 1. (This property rests on the fact that asymptotically—for a high discretization level of the multigrid—the number of all nodes grows by a factor of 4 from one level to the next higher one.)

In other words, too many nodes have obtained colour 1. The cheapest way then to obtain an asymptotically "uniform" overall colouring is to subdivide the old nodes into five colour sets and every colour set of the new nodes into three subsets. In this way, we arrive at an asymptotically "uniform" 20-colouring of the graph.

REMARK. Alternatively, the above theorem is not applied but the new nodes are 3-coloured. Though possibly expensive, this can be done (see the Remark to Proposition 3, but also the Theorem 3 below) and results into an asymptotically "uniform" 4-colouring: To each of the coarse-grid triangles we can assign 1/2 of its side midpoints (our new nodes), and $3 \cdot (1/6)$ of its corner points (our old nodes; most of them, on a high level, have degree 6)—i.e., one half of every colour. ∎

4. LINEAR-TIME 6-, 5- AND 4-COLOUR ALGORITHMS

In what follows we give several algorithms which satisfy both side conditions.

THEOREM 1. *The simple sequential algorithm (consisting in cycling through all nodes, always colouring the considered node if it is colourable; if there remain uncoloured nodes, then a next cycle with a new colour follows) satisfies both side conditions if the old nodes are taken first. This algorithm results into a 6-colouring for our special graphs.*

This result follows in a well-known way from the fact that the maximal degree of the graph of the new nodes is 4; we detail it for illustration:

After colouring the old nodes and deleting them and their connections to the new nodes there remains the graph of the new nodes which is of a rather special structure. If adding, at the

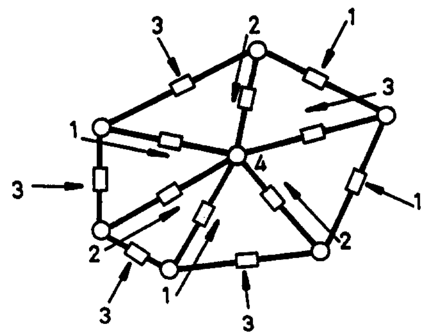

Figure 3.

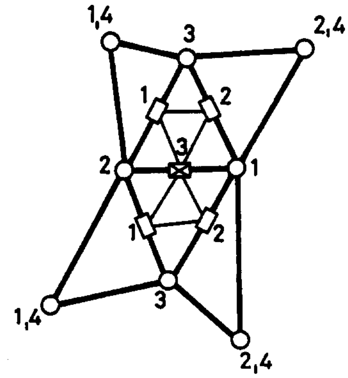

Figure 4.

boundary of the domain in which the boundary value problem has been posed, artificial connections between new nodes which have a common old node neighbour, then the remaining graph is a so-called Euler graph of degree 4, see Figure 2. (Every inner new node is, originally, of degree 6 since six triangles meet there; from these, its two connections to old nodes are deleted. At the boundary, new nodes are originally of degree 4, two connections to old nodes are deleted, two artificial connections to new nodes are added.)

That is, after colouring of the old nodes, every new node can be considered to be of degree 4. Hence, if after the next 4 colouring cycles he remains uncoloured, he is surrounded by coloured nodes before starting the sixth colouring cycle and therefore can be coloured in that cycle. ∎

REMARKS.

1. The 6-colouring so constructed is usually quite nonuniform, see 5.1.
2. The sequential 6-colour algorithm can be modified to a 5-colour algorithm by introduction of an inspection step after the 3. colouring cycle. It turns out that then only lines (which don't bifurcate), triangles and circles with even number of (new) nodes can remain uncoloured. From these, the only dangerous configurations are the uncoloured triangles— which have been observed experimentially also and can be broken up (partially coloured, using colours 2 and 3) employing local information only. Then a 4. and 5. cycle finishes the colouring. ∎

The algorithm of the next theorem is even simpler and also does not contain recursive steps like [19].

THEOREM 2. *Taking the old nodes first and the new nodes with the smallest last ordering next, a 5-colouring results.*

PROOF. The proof is analogous to that of Proposition 5: Consider a degree-2 node at the boundary of the graph of the new nodes. If its neighbours are coloured somehow, a third colour suffices to colour our node. Therefore we put the node into a list, delete it along with its edges from the graph, postponing the question how the neighbours got their colours. Similarly we put all degree-2 nodes into the list. Returning then to the neighbours of our node, they may turn out to be of degree 2 now. We repeat our argument and put them into the list etc. Continuing this way, it may happen that the list exhausts the graph. Then we colour its nodes in opposite order (and 3 colours suffice for this). Otherwise, there remain only nodes of order 3 and 4. Now we assume that the neighbours of a degree-3 node are coloured somehow. Then a fourth colour suffices to colour the considered node. Therefore we put now all degree-3 nodes into the list, deleting at the same time their edges from the graph. In this way further nodes of maximal degree 3 arise and the process can be continued until we exhaust the graph. Turning around the list, the nodes can be coloured using 4 colours (the colours 2 to 5) now. ∎

The preceding algorithms can be considered to apply to a fixed level (other than the first) of the multigrid method; we used only the properties of the graph originating from one refinement step of the triangles. Working in multigrid spirit, a 4-colour algorithm can be derived.

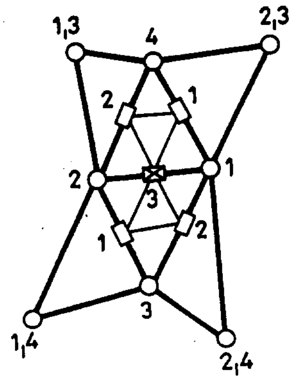

Figure 5.

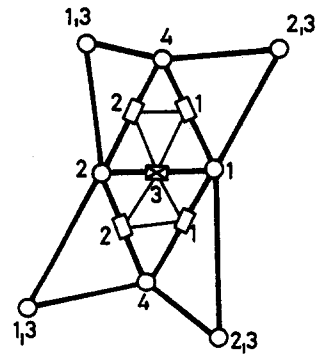

Figure 6.

THEOREM 3. *Assume, on the first level, that a 3- or 4-colouring is given. Then a 4-colouring on every of the following levels can be constructed from the previous one using the following 3 steps.*

1. *For all colour-4 points (if there are any), and on all edges starting from them, if the other endpoint of the edge is coloured i then assign i also to the midpoint—the new node of that edge ($i = 1, 2, 3$, see Figure 3).*
2. *For all edges without colour 4, if the nodes connected have colour i and j then the midpoint between them gets colour $6 - i - j$ (that colour which is different from i, j, 4).*
3. *The old nodes get colour 4 now.*

PROOF. It suffices to show, for a fixed new node and the two adjacent triangles, that no conflicts arise (between the colours of the graph of the new nodes) around the fixed new node through application of the steps 1 and 2. (In the following figures, ⊠ is the considered new node.) Observe that the midpoint colours of the two triangles are not influenced by colours outside of these triangles.

First case: in the considered two triangles adjacent to the new node no corner (no old node) is of colour 4. See Figure 4.
Second case: there is one colour-4 corner, Figure 5.
Third case: there are two colour-4 corners, Figure 6.

All other constellations follow by reflections or by cyclic permutation $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$. ∎

REMARKS.

1. Now the old nodes have got colour 4. This is not decisive; it is important only that they are taken first during the multigrid smoothing iteration.
2. The first and third case of the proof can arise only at the first level, when constructing the second level.
3. The remark at the end of Point 3 applies: The constructed colouring is asymptotically "uniform."
4. For this algorithm, the second side condition holds if the cost of constructing the first-level colouring is not counted. In other words, all the difficulty is shifted to the first level. However, it is the sense of the multigrid solution algorithm that on that level there are only a few nodes.
5. In linear time, a 5-colouring can be constructed on the first level [19, 20]. Then, performing essentially the same steps as before to construct the 4-colouring on the next levels, the rate of appearance of one of the coloures, say 5, can be diminished from level to level fastly by creating, to every old 5-coloured node, only two 5-coloured new nodes (instead of four). In this way, asymptotically, a uniform 4-colouring is approached. ∎

We observe that the three colouring algorithms considered are useful not only for the multigrid solution of discretized elliptic equations, but also for preconditioned conjugated gradient iteration.

To finish this point, we emphasize once more that we don't prefer the 4-colourings as compared to the 6- or 5-colourings; rather all three could be part of a set of colourings from which to select the one which is best suited to a given computing system. Further colourings can be derived from the considered ones by subdividing sets of equally coloured nodes, and this can help to equalize the numbers of nodes per colours.

5. COMPUTER EXPERIMENTS

5.1

Besides the rectangular cases mentioned in the Introduction there is one occasion of a symmetric grid which could give rise to very fast convergence if corresponding colouring is applied: a grid consisting of regular triangles.

Satisfying both side conditions, this grid is easily 4-coloured as follows: We select one of the systems of parallel straight lines which generate the grid. On those lines which contain the old nodes we use colours 1 and 2; on lines in between them the nodes are coloured with colours 3 and 4.

This has been coded for the case of a regular triangle as domain of the first boundary value problem for the Poisson equation and solved by finite element based full multigrid using 4 and 5 levels (i.e., with 325 or 1225 nodes on the last grid; we employed two Gauss-Seidel pre-smoothing steps, one post-smoothing step only if entering first time a new level, and W-cycles). For comparison, 4 other (nonsymmetric) colourings were applied, too. As restriction, "half injection" was selected [2, p. 78], as interpolation the linear finite element interpolation.

It turned out that the symmetric colouring produced error reduction factors (maximum norm of the old error divided by that of the new error) of maximally ~ 50 per iteration step on the last grid whereas the nonsymmetric colourings exhibited factors of maximally ~ 43. In a next series of experiments with restriction changed to the canonical one (adjoint of the interpolation), the results for symmetric colouring were not even slightly better as compared to the nonsymmetric colourings. As a matter of fact here all colourings produce rather fast convergence.

The nonsymmetric colourings were obtained by starting from the ordering of the inner grid nodes as obtained by the algorithm [22] designed for grid construction on general two-dimensional polygonal domains. If the sequential colouring algorithm of Theorem 1 is applied to this initial ordering, then the regular triangle with its 253 inner nodes on the fourth level is 6-coloured with 58, 60, 60, 33, 21, 21 nodes in the different colour sets. For the same algorithm, the typical distribution on the colour sets (as observed for 8 different domains, including symmetric, nonsymmetric, simple and multiple connected ones) is one with relatively yet more nodes in the second set, with relatively less nodes in the third set and with only exceptional nodes in sets 4 to 6.

5.2

In nonself-adjoint boundary value problems, orderings of the unknowns are known to be effective which are not regular colourings in the mentioned graph colouring sense: zebra-line ordering and streamline-wise ordering, see [1]. In experiments performed by the first author for diffusion-convection equations with dominant convection, streamwise orderings proved to be in the mean (not in every single case but over several problems and restriction/interpolation combinations) clearly better than 4-colour orderings.

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